

The Discrete and Continuous Markus–Yamabe Stability Conjectures

Álvaro Castañeda and Víctor Guíñez *

Abstract

We study the discrete and continuous versions of the Markus–Yamabe Conjecture for polynomial vector fields in \mathbb{R}^n (especially when $n = 3$) of the form $X = \lambda I + H$ where λ is a real number, I the identity map, and H a map with nilpotent Jacobian matrix JH . We consider the case where the rows of JH are linearly dependent over \mathbb{R} and that where they are linearly independent over \mathbb{R} . In the former, we find non-linearly triangularizable vector fields X for which the origin is a global attractor for both the continuous and the discrete dynamical systems generated by X . In the independent continuous case, we present a family of vector fields which have orbits escaping to infinity. In the independent discrete case, we present a large family of vector fields which have a periodic point of period 3.

1 Introduction

Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a C^1 –vector field with $F(p) = 0$. Consider the differential system

$$\dot{x} = F(x). \quad (1)$$

We let $\phi(t, x)$ denote the solution of (1) with initial condition $\phi(0, x) = x$. We say that p is a *global attractor* of the differential system (1) if for each $x \in \mathbb{R}^n$, we have that $\phi(t, x)$ is defined for all $t > 0$ and tends to p as t tends to infinity.

*The authors were supported in part by FONDECYT Grant #1080172, by CONICYT Grant PBCT ADI 17, and by MATH-AMSUD DySET. The first author was also supported by FONDECYT Postdoctoral Grant #3100082 and Mecesp PUC-0711.

In [MY], L. Markus and H. Yamabe establish their well known global stability conjecture.

The Markus–Yamabe Conjecture (MYC). Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a C^1 –vector field with $F(0) = 0$. If for any $x \in \mathbb{R}^n$ all the eigenvalues of the Jacobian of F at x have negative real part, then the origin is a global attractor of the differential system (1).

The corresponding version of the MYC for discrete dynamical systems is as follows. Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a C^1 –vector field. Consider the sequence

$$x^{(m+1)} = F(x^{(m)}), \quad x^{(0)} \in \mathbb{R}^n. \quad (2)$$

Consider also the dynamics of the iterations of F . Let p be a fixed point of F , that is, $F(p) = p$. We say that p is a *global attractor* of the discrete dynamical system (2) if the sequence $x^{(m)}$ tends to p as m tends to infinity, for any $x^{(0)} \in \mathbb{R}^n$.

The Discrete Markus–Yamabe Conjecture (DMYC). Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a C^1 –vector field with $F(0) = 0$. If for any $x \in \mathbb{R}^n$ all the eigenvalues of the Jacobian of F at x have modulus less than one, then the origin is a global attractor of the discrete dynamical system (2) generated by F .

It is known that the MYC (resp. the DMYC) is true when $n \leq 2$ (resp. $n = 1$) and false when $n \geq 3$ (resp. $n \geq 2$), though both conjectures are true for triangular vector fields in any dimension. For these vector fields, L. Markus and H. Yamabe prove the continuous case in [MY], and A. Cima et al. prove the discrete case in [CGM2]. For polynomial vector fields, the DMYC is also true when $n = 2$ (see [CGM2]), though both conjectures are false when $n \geq 3$. For an example of a pair of polynomial vector fields, of which one satisfies the MYC hypotheses and the other the DMYC hypotheses, having both vector fields orbits that escape to infinity, see [CEGMH]. Further in [CGM1], A. Cima et al. obtain a family of polynomial counterexamples containing the preceding pair.

In this paper we study both conjectures in the case of a special family of polynomial vector fields in \mathbb{R}^n , focusing on $n = 3$. Given a real number λ and a positive integer n , we denote the set consisting of the polynomial vector fields in \mathbb{R}^n of the form $F = \lambda I + H$, where I is the identity map and H has nilpotent Jacobian matrix at every point, by $\mathcal{N}(\lambda, n)$. Note that for this class of vector fields, the Jacobian matrix at each $x \in \mathbb{R}^n$ has all its eigenvalues

equal to λ . Therefore, a vector field $F = \lambda I + H$ in $\mathcal{N}(\lambda, n)$ satisfies the MYC (resp. the DMYC) hypotheses if and only if $\lambda < 0$ (resp. $|\lambda| < 1$). The counterexamples of [CGM1] are, basically, vector fields $X = \lambda I + H$ in $\mathcal{N}(\lambda, 3)$ where H is a quasi-homogeneous vector field of degree one. In [Gui-Cast], we give examples of vector fields in $\mathcal{N}(\lambda, 3)$ which are linearly triangularizable (that is, triangular after a linear change of coordinates). For these vector fields, the MYC (resp. the DMYC) is true when $\lambda < 0$ (resp. $|\lambda| < 1$). Further, the paper contains a family of counterexamples to the MYC which generalizes that of Cima–Gasull–Mañosas.

Polynomial vector fields H defined on \mathbb{R}^n and on \mathbb{C}^n with nilpotent Jacobian matrix at every point have been extensively studied from the algebraic geometry viewpoint (see for example [vE]). In this paper we make use of some aspects of this theory.

The examples and counterexamples $X = \lambda I + H \in \mathcal{N}(\lambda, n)$ of above have one common characteristic, namely the rows of JH are linearly dependent over \mathbb{R} . Thus we are led to introducing the sets $\mathcal{N}_{ld}(\lambda, n)$ and $\mathcal{N}_{li}(\lambda, n)$. The first is the set consisting of the polynomial vector fields $X = \lambda I + (H_1, \dots, H_n)$ in $\mathcal{N}(\lambda, n)$ such that $\{H_1, \dots, H_n\}$ is linearly dependent over \mathbb{R} . The second set is $\mathcal{N}_{li}(\lambda, n) = \mathcal{N}(\lambda, n) - \mathcal{N}_{ld}(\lambda, n)$. Section 2 studies the linearly dependent case, especially when the dimension is three. We give a normal form for the vector fields of $\mathcal{N}_{ld}(\lambda, 3)$ (see Proposition 2.2) and characterize those elements which are linearly triangularizable (see Theorem 2.9). The normal form depends on a polynomial $f(t)$ with coefficients in $\mathbb{R}[z]$. In the case $f(t)$ is a polynomial of degree one, we show that the corresponding vector fields satisfy both conjectures (see Theorems 2.10 and 2.11). We thus obtain a family of non-linearly triangularizable vector fields in $\mathcal{N}_{ld}(\lambda, 3)$ for which both conjectures are true. To our knowledge, there are no examples as the preceding one in the literature. In the case the degree of $f(t)$ is greater than one, we give a new family of counterexamples to both the MYC and the DMYC (see Proposition 2.12). The foregoing considerations lead us to raising the question, Do there exist vector fields in $\mathcal{N}_{ld}(\lambda, 3)$ with the degree of $f(t)$ greater than one for which the MY Conjecture, or the DMY Conjecture, or both, are true? The section concludes showing that, for a vector field $X \in \mathcal{N}_{ld}(\lambda, 3)$, in order for the origin not to be a global attractor the vector field must have at least one orbit which escapes to infinity (see Theorem 2.14).

In Section 3 we deal with the linearly independent case. We state the Dependence Problem and the Generalized Dependence Problem introduced

by A. van den Essen in [vE, Chapter 7], among others, and we obtain a family of examples $F_{n,r} = \lambda I + H_{n,r}$ in $\mathcal{N}_{li}(\lambda, n)$ for any dimension $n \geq 3$, with $\text{rk } JH_{n,r} = r \geq 2$. When $n \geq 3$ and $r = 2$, we show that both conjectures are false for these vector fields (see Theorem 3.2). Subsequently, we consider vector fields $X = \lambda I + H \in \mathcal{N}_{li}(\lambda, 3)$, where $H(x, y, z) = (u(x, y, z), v(x, y, z), h(u(x, y, z), v(x, y, z)))$. For a characterization of a large class of these vector fields H , see [ChE]. The characterization depends on a polynomial map $g(t)$. In the case $g(t)$ is a polynomial of degree less than or equal to two, we show that the vector field $X = \lambda I + H$, with $\lambda < 0$, has orbits that escape to infinity (see Theorem 3.5). On the other hand, in the discrete case, for $|\lambda| < 1$, these maps have a periodic point of period three (see Theorem 3.9). Therefore, the DMYC is false for this class of maps. This scenario leads to posing the following question. Do there exist vector fields in $\mathcal{N}_{li}(\lambda, 3)$ for which the MY Conjecture, or the DMY Conjecture, or both, are true?

2 The linearly dependent case

Given a linear isomorphism $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and a vector field $F = \lambda I + H \in \mathcal{N}(\lambda, n)$, we have

$$T_*F = T \circ F \circ T^{-1} = \lambda I + T \circ H \circ T^{-1} \in \mathcal{N}(\lambda, n) \quad (3)$$

which implies that the set $\mathcal{N}(\lambda, n)$ is invariant by linear changes of coordinates as both a continuous dynamical system and a discrete dynamical system. Moreover, the vector field obtained after a linear change of coordinates is the same in both the continuous and the discrete cases.

Let $\mathcal{N}_{ld}(\lambda, n)$ be the set consisting of the polynomial vector fields $X = \lambda I + (H_1, \dots, H_n)$ in $\mathcal{N}(\lambda, n)$ such that $\{H_1, \dots, H_n\}$ is linearly dependent over \mathbb{R} . Let $\mathcal{N}_{li}(\lambda, n) = \mathcal{N}(\lambda, n) - \mathcal{N}_{ld}(\lambda, n)$. Note that the examples and counterexamples of [Gui-Cast] belong to the set $\mathcal{N}_{ld}(\lambda, 3)$. The following gives properties of these sets.

Proposition 2.1. *1) The sets $\mathcal{N}_{ld}(\lambda, n)$ and $\mathcal{N}_{li}(\lambda, n)$ are invariant by linear changes of coordinates.*

2) Let $X = \lambda I + (H_1, \dots, H_n) \in \mathcal{N}(\lambda, n)$ be such that $X(0) = 0$. Then $X \in \mathcal{N}_{ld}(\lambda, n)$ if and only if the rows of the Jacobian matrix $J(H_1, \dots, H_n)$ are linearly dependent over \mathbb{R} .

3) Let $X \in \mathcal{N}_{id}(\lambda, n)$. Then there exists a $T \in Gl_n(\mathbb{R})$ such that $T_*X = \lambda I + (H_1, \dots, H_{n-1}, 0)$. Moreover, for any $a \in \mathbb{R}$, the vector field $X_a : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$ defined by $X_a(x_1, \dots, x_{n-1}) = \lambda(x_1, \dots, x_{n-1}) + (H_1, \dots, H_{n-1})((x_1, \dots, x_{n-1}, a))$ belongs to $\mathcal{N}(\lambda, n-1)$.

Proof. Assertion 1) is clear. Assertion 2) follows from [vE, Exercise 7.1.1]. Concerning assertion 3), let $X = \lambda I + (G_1, \dots, G_n) \in \mathcal{N}_{id}(\lambda, n)$, and let $(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n - \{(0, \dots, 0)\}$ be such that $\alpha_1 G_1 + \dots + \alpha_n G_n \equiv 0$. Without loss of generality, we may suppose $\alpha_n \neq 0$. Consider the linear change of coordinates

$$(x_1, \dots, x_n) = T(u_1, \dots, u_n) = (u_1, \dots, u_{n-1}, \alpha_1 u_1 + \dots + \alpha_n u_n).$$

Then $T_*X = \lambda I + (H_1, \dots, H_{n-1}, 0)$ and the assertion follows easily. \square

Our next result gives a normal form for the vector fields in $\mathcal{N}_{id}(\lambda, 3)$. For a proof, see for example [ChE, Corollary 1.1].

Proposition 2.2. *Let $X = \lambda I + (S, U, V) \in \mathcal{N}_{id}(\lambda, 3)$. Then there exists a $T \in Gl_3(\mathbb{R})$ such that $T_*X = \lambda I + (P, Q, 0)$ where*

$$\begin{aligned} P(x, y, z) &= -b(z)f(a(z)x + b(z)y) + c(z) \quad \text{and} \\ Q(x, y, z) &= a(z)f(a(z)x + b(z)y) + d(z) \end{aligned} \tag{4}$$

with $a, b, c, d \in \mathbb{R}[z]$ and $f \in \mathbb{R}[z][t]$.

Remark 2.3. *In the normal form (4) we may assume $f(0) = 0$ by modifying the polynomials $c(z)$ and $d(z)$ if necessary.*

An interesting question about the vector fields satisfying the hypotheses of the MYC or the DMYC concerns the injectivity.

Proposition 2.4. *Any $X \in \mathcal{N}_{id}(\lambda, 3)$ is injective.*

Proof. The Proposition results from the normal form (4). \square

Consider the following sets. Let $\mathcal{N}_{CY}(\lambda, n)$ (resp. $\mathcal{N}_{DY}(\lambda, n)$) be the subset of $\mathcal{N}(\lambda, n)$ consisting of the polynomial vector fields X such that the origin is a global attractor for the differentiable system $\dot{x} = X(x)$ (resp. the discrete dynamical system generated by X).

Recall that a vector field $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is *triangular* if it has the form

$$F(x_1, x_2, \dots, x_n) = (F_1(x_1), F_2(x_1, x_2), \dots, F_n(x_1, x_2, \dots, x_n)).$$

Let $\mathcal{N}_{CT}(\lambda, n)$ (resp. $\mathcal{N}_{DT}(\lambda, n)$) be the subset of $\mathcal{N}(\lambda, n)$ consisting of the polynomial vector fields F for which there exists a diffeomorphism $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that G_*F (resp. $G \circ F \circ G^{-1}$) is triangular. The elements of $\mathcal{N}_{CT}(\lambda, n)$ (resp. $\mathcal{N}_{DT}(\lambda, n)$) are called *triangularizable*. An element F in $\mathcal{N}_{CT}(\lambda, n)$ (resp. $\mathcal{N}_{DT}(\lambda, n)$) is said to be *linearly triangularizable* if there exists a linear change of coordinates which makes F triangular. We denote the set consisting of the vector fields in $\mathcal{N}(\lambda, n)$ which are linearly triangularizable by $\mathcal{N}_{LT}(\lambda, n)$. Since the MYC is true for C^1 -vector fields in dimension two and the DMYC is true for polynomial vector fields also in dimension two, and since both conjectures are true for triangular vector fields in any dimension, we have the following.

Theorem 2.5. *Let $\lambda \in \mathbb{R}$, and let $n \geq 2$ be an integer.*

Then:

- a) *If $\lambda < 0$ (resp. $|\lambda| < 1$), then $\mathcal{N}(\lambda, 2) = \mathcal{N}_{CY}(\lambda, 2)$ (resp. $\mathcal{N}(\lambda, 2) = \mathcal{N}_{DY}(\lambda, 2)$).*
- b) *If $\lambda < 0$, then $\mathcal{N}_{LT}(\lambda, n) \subset \mathcal{N}_{CT}(\lambda, n) \subset \mathcal{N}_{CY}(\lambda, n) \cap \mathcal{N}_{Id}(\lambda, n)$.*
- c) *If $0 < |\lambda| < 1$, then $\mathcal{N}_{LT}(\lambda, n) \subset \mathcal{N}_{DT}(\lambda, n) \subset \mathcal{N}_{DY}(\lambda, n) \cap \mathcal{N}_{Id}(\lambda, n)$.*

Proof. For a proof of assertion a) in the continuous case, see for example C. Gutierrez [Gu]; in the discrete case, see [CGM2, Theorem B]. Assertion b) follows from [MY, Theorem 4]. Assertion c) follows from [CGM2, Theorem A]. \square

Using the preceding notation, the main result of [Gui-Cast, Section 2] and its corollaries can be rewritten as follows.

Theorem 2.6. *Let $\lambda \in \mathbb{R}$, and let $m \geq 1$ be an integer. Assume $X_k = \lambda I + H_1 + \dots + H_k \in \mathcal{N}_{Id}(\lambda, 3)$ for $1 \leq k \leq m$, where H_i is a homogeneous polynomial of degree i , with $i = 1, \dots, m$. Then $X_k \in \mathcal{N}_{LT}(\lambda, 3)$ and*

$$X_m(x, y, z) = \lambda(x, y, z) + (0, a_1 x + \dots + a_m x^m, r_1(x, y) + \dots + r_m(x, y))$$

where $r_i(x, y)$ is a homogeneous polynomial of degree i , with $1 \leq i \leq m$, up to a linear change of coordinates.

Corollary 2.7. *Let $\lambda \in \mathbb{R}$, and let k, m be integers such that $1 \leq k < m$. Any polynomial vector field $X = \lambda I + H_k + H_m \in \mathcal{N}(\lambda, 3)$, with H_k and H_m homogeneous polynomials of degree k and m , respectively, belongs to $\mathcal{N}_{LT}(\lambda, 3)$.*

Corollary 2.8. *Let $\lambda \in \mathbb{R}$. Any polynomial vector field $X = \lambda I + H_2 + H_3 \in \mathcal{N}(\lambda, 3)$, with H_2 and H_3 homogeneous polynomials of degree 2 and 3, respectively, belongs to $\mathcal{N}_{LT}(\lambda, 3)$.*

Our next result establishes conditions under which vector fields of the form $X = \lambda I + (P, Q, 0) \in \mathcal{N}_{ld}(\lambda, 3)$, with (P, Q) as in Proposition 2.2, are linearly triangularizable.

Theorem 2.9. *Let $X = \lambda I + H \in \mathcal{N}_{ld}(\lambda, 3)$ where*

$$H(x, y, z) = f(a(z)x + b(z)y)(-b(z), a(z), 0) + (c(z), d(z), 0)$$

with $\lambda \in \mathbb{R}$, $a, b, c, d \in \mathbb{R}[z]$, $f \in \mathbb{R}[z][t]$, and $X(0) = 0$. Then $X \in \mathcal{N}_{LT}(\lambda, 3)$ if and only if either f is constant or $\{a, b\}$ are linearly dependent over \mathbb{R} .

Proof. When f depends only on z , the result is clear. In what follows, we will assume that the degree of f with respect to t is greater than zero. If $\{a, b\}$ are linearly dependent over \mathbb{R} , then there exists $(\alpha, \beta) \in \mathbb{R}^2 - \{(0, 0)\}$ such that $\alpha a(z) + \beta b(z) = 0$, for all $z \in \mathbb{R}$. Assume $\beta \neq 0$. Then $b(z) = \delta a(z)$, with $\delta = -\frac{\alpha}{\beta}$. Consider the linear isomorphism $T(x, y, z) = (z, x + \delta y, y)$. Then

$$T_*(X)(u, v, w) = \lambda(u, v, w) + (0, c(u) + \delta d(u), a(u)f(a(u)v) + d(u)),$$

which is triangular.

Now suppose that there exists a linear isomorphism M such that

$$M_*(X)(u, v, w) = \lambda(u, v, w) + (A(v, w), B(w), 0).$$

Assume that $[M] = (m_{ij})_{1 \leq i, j \leq 3}$ is the matrix of M with respect to the canonical basis of \mathbb{R}^3 . We have

$$m_{31}[-f(t)b(z) + c(z)] + m_{32}[f(t)a(z) + d(z)] \equiv 0$$

where $t = a(z)x + b(z)y$. Then

$$m_{31}[-f(0)b(z) + c(z)] + m_{32}[f(0)a(z) + d(z)] \equiv 0$$

and, therefore,

$$(f(t) - f(0)) [-m_{31} b(z) + m_{32} a(z)] \equiv 0.$$

If $(m_{31}, m_{32}) \neq (0, 0)$, the proof is complete. If $(m_{31}, m_{32}) = (0, 0)$, we may assume that $m_{33} = 1$ and $\det[M] = 1$. Thus the matrix of M^{-1} with respect to the canonical basis of \mathbb{R}^3 is

$$[M^{-1}] = \begin{pmatrix} m_{22} & -m_{12} & \tilde{m}_{13} \\ -m_{21} & m_{11} & \tilde{m}_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

with $\tilde{m}_{13} = -m_{13} m_{22} + m_{12} m_{23}$ and $\tilde{m}_{23} = m_{13} m_{21} - m_{11} m_{23}$. Then

$$t = a(w) [m_{22} u - m_{12} v + \tilde{m}_{13} w] + b(w) [-m_{21} u + m_{11} v + \tilde{m}_{23} w]$$

and

$$B(w) = m_{21} [-f(t) b(w) + c(w)] + m_{22} [f(t) a(w) + d(w)].$$

Differentiating the preceding expression with respect to u we obtain

$$0 = f'(t) [m_{22} a(w) - m_{21} b(w)]^2$$

and so $\{a, b\}$ are linearly dependent over \mathbb{R} , which completes the proof. \square

The next two results assert that, in the linearly dependent case, the origin is always a global attractor when the degree of the polynomial $f(t)$ is one. These results give examples of vector fields in $\mathcal{N}_{CY}(\lambda, 3)$ and in $\mathcal{N}_{DY}(\lambda, 3)$ which are not linearly triangularizable. By Remark 2.3, it suffices to consider the case $f(t) = g(z) t$.

Theorem 2.10. *Let $X = \lambda I + H \in \mathcal{N}_{ld}(\lambda, 3)$ where*

$$H(x, y, z) = g(z) (a(z) x + b(z) y) (-b(z), a(z), 0) + (c(z), d(z), 0)$$

with $\lambda < 0$, $a, b, c, d, g \in \mathbb{R}[z]$ and $X(0) = 0$. Then $X \in \mathcal{N}_{CY}(\lambda, 3)$.

Proof. Note that $(x(t), y(t), z(t))$ is a solution of the differential system $\dot{x} = X(x)$ if and only if $z(t) = z_0 e^{\lambda t}$ and $(x(t), y(t))$ is a solution of the linear system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \lambda - A(t)B(t)G(t) & -B(t)^2G(t) \\ A(t)^2G(t) & \lambda + A(t)B(t)G(t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} C(t) \\ D(t) \end{pmatrix}$$

where $(A, B, C, D, G)(t) = (a, b, c, d, g)(z_0 e^{\lambda t})$. Since the origin is a local attractor, there is a basis of solutions of the linear system consisting of solutions which tend to the origin as t tends to $+\infty$. Therefore, the origin is a global attractor. \square

Theorem 2.11. *Let $F = \lambda I + H \in \mathcal{N}_{ld}(\lambda, 3)$ where*

$$H(x, y, z) = g(z) (a(z)x + b(z)y) (-b(z), a(z), 0) + (c(z), d(z), 0)$$

with $0 < \lambda < 1$, $a, b, c, d, g \in \mathbb{R}[z]$, and $X(0) = 0$. Then $F \in \mathcal{N}_{DY}(\lambda, 3)$.

Proof. Without loss of generality, we may assume that $c(z) \equiv d(z) \equiv 0$. In fact, the polynomials $c(z)$ and $d(z)$ may be eliminated by applying the coordinate change $T(u, v, w) = (u + m(w), v + n(w), w)$ where

$$\begin{pmatrix} m(w) \\ n(w) \end{pmatrix} = \frac{-1}{(1-\lambda)^2} [(1-\lambda)I + g(w)M(w)] \begin{pmatrix} c(w) \\ d(w) \end{pmatrix}$$

with

$$M(w) = \begin{pmatrix} -a(w)b(w) & -b(w)^2 \\ a(w)^2 & a(w)b(w) \end{pmatrix}.$$

So we assume

$$H(x, y, z) = g(z) (a(z)x + b(z)y) (-b(z), a(z), 0).$$

Therefore,

$$F(x, y, z) = (A(z) \begin{pmatrix} x \\ y \end{pmatrix}, \lambda z)$$

where

$$A(z) = \begin{pmatrix} \lambda - a(z)b(z)g(z) & -b(z)^2g(z) \\ a(z)^2g(z) & \lambda + a(z)b(z)g(z) \end{pmatrix}.$$

Thus it suffices to prove that, for any $(x, y) \in \mathbb{R}^2$, we have

$$\lim_{n \rightarrow \infty} A(\lambda^n z) A((\lambda^{n-1} z)) \dots A(\lambda z) A(z) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Let \mathcal{M}_2 be the normal vector space of the 2×2 real matrices $A = (a_{ij})$ endowed with the norm $\|A\| = 2 \max |a_{ij}|$. Considering \mathbb{R}^2 endowed with the norm $\|(x, y)\| = \max\{|x|, |y|\}$, we have

$$\|A((x, y))\| \leq \|A\| \|(x, y)\| \quad \text{and} \quad \|AB\| \leq \|A\| \|B\|.$$

A simple computation yields

$$A(\lambda^n z)A((\lambda^{n-1} z) \dots A(\lambda z)A(z) =$$

$$\begin{pmatrix} \lambda^n - na_0 b_0 g_0 \lambda^{n-1} + r_{11}(z) & -nb_0^2 g_0 \lambda^{n-1} + r_{12}(z) \\ na_0^2 g_0 \lambda^{n-1} + r_{21}(z) & \lambda^n + na_0 b_0 g_0 \lambda^{n-1} + r_{22}(z) \end{pmatrix}$$

where $r_{ij}(0) = 0$ and $(a_0, b_0, g_0) = (a, b, g)(0)$.

Fix $N \in \mathbb{N}$ so that $2N\lambda^{N-1} \max\{a_0^2 g_0, b_0^2 g_0, |a_0 b_0 g_0|\} < 1$. Let $B(z) = A(\lambda^{N-1} z)A((\lambda^{N-2} z) \dots A(\lambda z)A(z)$. Consider $0 < |z| < z_0$ such that $\|B(z)\| \leq K < 1$. Then, for $n = kN - 1$, we have

$$\left\| A(\lambda^n z) \dots A(z) \begin{pmatrix} x \\ y \end{pmatrix} \right\| = \left\| B(\lambda^{(k-1)N} z) \dots B(z) \begin{pmatrix} x \\ y \end{pmatrix} \right\|$$

$$\leq K^k \|(x, y)\| \rightarrow 0 \quad \text{if } k \rightarrow \infty$$

which completes the proof. \square

When the degree of the polynomial $f(t)$ is greater than one, there are examples of polynomial vector fields in $\mathcal{N}_{ld}(\lambda, 3)$ having orbits that escape to infinity.

Proposition 2.12. *Let $F = \lambda I + H \in \mathcal{N}_{ld}(\lambda, 3)$ where*

$$H(x, y, z) = z^{k-1} (x + yz)^{k+1} (-z, 1, 0).$$

Then:

a) *If $\lambda < 0$ and k odd, then $F \notin \mathcal{N}_{CY}(\lambda, 3)$.*

b) *If $0 < |\lambda| < 1$, then $F \notin \mathcal{N}_{DY}(\lambda, 3)$.*

Proof. a) By applying the coordinate change

$$(u, v, w) = T(x, y, z) = (z(x + yz), \lambda yz^2, z) \tag{5}$$

outside of $z = 0$, we obtain

$$T_*(F)(u, v, w) = (2\lambda u + v, 3\lambda v + \lambda u^{k+1}, \lambda w). \tag{6}$$

The planar system

$$(\dot{u}, \dot{v}) = (2\lambda u + v, 3\lambda v + \lambda u^{k+1}) \quad (7)$$

has two singular points, namely the origin and the point $u_0(1, -2\lambda)$ where $u_0 = \sqrt[k]{6\lambda}$. For $z_0 \neq 0$, the latter singular point generates the following orbit of the original system

$$\gamma(t) = \left(\frac{3 \sqrt[k]{6\lambda}}{z_0} e^{-\lambda t}, -\frac{2 \sqrt[k]{6\lambda}}{z_0^2} e^{-2\lambda t}, z_0 e^{\lambda t} \right)$$

which escapes to infinity as t tends to $+\infty$.

b) Again, by applying the coordinate change (5) outside of $z = 0$, we obtain

$$T \circ F \circ T^{-1}(u, v, w) = \lambda(\lambda u + (\lambda - 1)(v + u^{k+1}), \lambda^2(v + u^{k+1}), w). \quad (8)$$

The planar map

$$(u, v) \rightarrow \lambda(\lambda u + (\lambda - 1)(v + u^{k+1}), \lambda^2(v + u^{k+1}))$$

has two fixed points, namely the origin and the point $u_0(1, -\lambda^2(1 + \lambda))$ where $u_0 = \sqrt[k]{\frac{(1+\lambda)(\lambda^3-1)}{\lambda}}$. For $z_0 \neq 0$, the latter fixed point generates the orbit of the original map

$$(x_n, y_n, z_n) = \left(\frac{(1 + \lambda + \lambda^2)u_0}{\lambda^n z_0}, -\frac{(1 + \lambda)u_0}{\lambda^{2n-1} z_0^2}, \lambda^n z_0 \right)$$

which escapes to infinity as n tends to $+\infty$. \square

Remark 2.13. *Assertion a) of Proposition 2.12 is also true if we consider $n \geq 2$, $a(z) = 1$, $b(z) = z$, and $f(t) = z^{n-2}(A_1(z)t + \dots + A_n(z)t^n)$, with n either even and $A_n(0) \neq 0$ or n odd and $A_n(0) < 0$.*

Thus we are led to posing the following.

Question 1. Do there exist vector fields in $\mathcal{N}_{ld}(\lambda, 3)$, with the degree of $f(t)$ greater than one, for which either the MY Conjecture, or the DMY Conjecture, or both, are true?

Our next result shows that, for a vector field $X \in \mathcal{N}_{ld}(\lambda, 3)$, in order for the origin not to be a global attractor the vector field must have at least one orbit which escapes to infinity.

Theorem 2.14. *Let $\lambda \in \mathbb{R}$, and let $F = \lambda I + H \in \mathcal{N}_{id}(\lambda, 3)$. If $\lambda < 0$ (resp. $|\lambda| < 1$) and $F \notin \mathcal{N}_{CY}(\lambda, 3)$ (resp. $F \notin \mathcal{N}_{DY}(\lambda, 3)$), then the differential system $\dot{x} = F(x)$ (resp. discrete dynamical system generated by F) has orbits which escape to infinity.*

Proof. We may assume that $H = (P, Q, 0)$ where

$$\begin{aligned} P(x, y, z) &= -b(z) f(a(z)x + b(z)y) + c(z) \quad \text{and} \\ Q(x, y, z) &= a(z) f(a(z)x + b(z)y) + d(z) \end{aligned}$$

with $a, b, c, d \in \mathbb{R}[z]$ and $f \in \mathbb{R}[z][t]$.

Consider the case $\lambda < 0$. Let $\gamma(t) = (x(t), y(t), z(t))$ be a solution of the system $\dot{x} = F(x)$. We denote the omega-limit set of γ by $\omega(\gamma)$. Since $z(t) = z(0)e^{\lambda t}$, we have $\omega(\gamma) \subset W_\infty$, where W_∞ is the extended plane $\{z = 0\} \cup \{\infty\}$. If the orbit $\gamma(t)$ is bounded, then $\omega(\gamma) = \{0\}$, thus obtaining the theorem. The proof for the discrete dynamical system generated by F is analogous. □

3 The linearly independent case

In this section we consider the linearly independent case. We begin with some algebraic preliminaries extracted from [vE, Chapter 7] and [ChE]. The study of the Jacobian Conjecture for polynomial maps of the form $I + H$, where I is the identity map and H a homogeneous map of degree 3, with JH nilpotent, led various authors to the following problem. Let κ be a field of characteristic zero.

Dependence Problem. Let $d \in \mathbb{N}$, with $d \geq 1$, and let $H = (H_1, \dots, H_n) : \kappa^n \rightarrow \kappa^n$ be a homogeneous polynomial map of degree d such that JH is nilpotent. Does it follow that H_1, \dots, H_n are linearly dependent over κ ?

The attempt to solve it by induction led to consider the more general problem:

Generalized Dependence Problem. Let $H = (H_1, \dots, H_n) : \kappa^n \rightarrow \kappa^n$ be a polynomial map such that JH is nilpotent. Are the rows of JH linearly dependent over κ ?

The answer to this question turned out to be “yes” if $n \leq 2$ and “no” if $n \geq 3$. More precisely, the following is proved by van den Essen in [vE, Theorem 7.1.7].

Theorem 3.1. *i) If JH is nilpotent and $\text{rk } JH \leq 1$, then the rows of JH are linearly dependent over κ (here rk is the rank as an element of $M_n(\kappa(X))$).*

ii) Let $r \geq 2$. Then, for any dimension $n \geq r+1$, there exists a polynomial map $H_{n,r} : \kappa^n \rightarrow \kappa^n$ such that $JH_{n,r}$ is nilpotent, $\text{rk } JH_{n,r} = r$, and the rows of $JH_{n,r}$ are linearly independent over κ .

As an example, let $a \in \mathbb{R}[x_1]$, with $\deg a = r$ and $f(x_1, x_2) = x_2 - a(x_1)$. Then $H_{n,r} = (H_1, \dots, H_n)$, where

$$\begin{aligned} H_1(x_1, \dots, x_n) &= f(x_1, x_2), \\ H_i(x_1, \dots, x_n) &= x_{i+1} + \frac{(-1)^i}{(i-1)!} a^{(i-1)}(x_1) (f(x_1, x_2))^{i-1}, \text{ if } 2 \leq i \leq r, \\ H_{r+1}(x_1, \dots, x_n) &= \frac{(-1)^{r+1}}{r!} a^{(r)}(x_1) (f(x_1, x_2))^r, \text{ and} \\ H_j(x_1, \dots, x_n) &= (f(x_1, x_2))^{j-1}, \text{ if } r+1 < j \leq n, \end{aligned}$$

is a polynomial map satisfying assertion ii). (See [vE, Proposition 7.1.9]). For $r = 2$ and $n \geq 3$, the components of $H_{n,2}$ are

$$\begin{aligned} H_1(x_1, \dots, x_n) &= x_2 - a x_1 - b x_1^2, \\ H_2(x_1, \dots, x_n) &= x_3 + (a + 2b x_1) (x_2 - a x_1 - b x_1^2), \\ H_3(x_1, \dots, x_n) &= -b (x_2 - a x_1 - b x_1^2)^2, \text{ and for } j \geq 4 \\ H_j(x_1, \dots, x_n) &= (x_2 - a x_1 - b x_1^2)^{j-1}, \end{aligned} \tag{9}$$

with $b \neq 0$.

Theorem 3.2. *Let $F_{n,2} = \lambda I + (H_1, \dots, H_n)$, with H_i as in (9).*

- a) If $\lambda < 0$, then the system $\dot{x} = F_{n,2}(x)$ has orbits that escape to infinity.*
- b) If $-1 < \lambda < 1$, then the discrete dynamical system generated by $F_{n,2}$ has a periodic orbit of period three.*

Proof. For assertion a), it suffices to prove the Theorem when $n = 3$. Set $X = F_{3,2}$. Then by applying the coordinate change $(u, v, w) = \phi(x_1, x_2, x_3) = b(x_1, x_3 - \lambda b x_1^2, \lambda x_1 + x_2 - a x_1 - b x_1^2)$, we have $\phi^*(X) = Y$ where

$$Y(u, v, w) = (w, \lambda v - w^2, 2\lambda w + v - \lambda^2 u).$$

We make the change of coordinates

$$(s, q, p) = \frac{1}{v}(1, u, w)$$

to find those orbits of Y which escape to infinity. If Z is the vector field Y in the new coordinates, then $W = (W_1, W_2, W_3) = sZ$ is defined by

$$W(s, q, p) = (-s(\lambda s - p^2), s(p - \lambda q) + qp^2, s(\lambda p + 1 - \lambda^2 q) + p^3).$$

For $s \neq 0$, the orbits of Z and W are the same. Moreover, for $s > 0$ (resp. $s < 0$), the orbits of Z and W have the same (resp. inverse) orientation. Over the plane $s = 0$, the vector field W is radially repelling outside of a line of singular points, namely the line $p = 0$. For $s > 0$, we have $W_1 > 0$ and, therefore, there are no orbits there with ω -limit set contained at $s = 0$. For $s < 0$, we must find orbits of W with α -limit set contained at $s = 0$. Indeed, consider the numbers

$$A = 2\lambda, \quad s_0 = \frac{1}{512\lambda^3}, \quad p_0 = -\frac{1}{8\lambda}, \quad q_0 = \frac{11}{16\lambda^2}$$

and the set

$$P_A = \{(s, q, p) : As - p^2 \leq 0, s_0 \leq s \leq 0, 0 \leq q \leq q_0, 0 \leq p \leq p_0\}.$$

We find:

- 1) Over the set $P_A \cap \{(s, q, p) : As - p^2 = 0\}$, the vector field W points outward from the set P_A . In fact, if $(s, q, p) \in P_A$ and $As - p^2 = 0$, then

$$\begin{aligned} AW_1 - 2pW_3 &= -\frac{p^3}{A}[p(A + 3\lambda) + 2(1 - \lambda^2 q)] \\ &\geq -\frac{p^3}{A}[p_0(A + 3\lambda) + 2(1 - \lambda^2 q_0)] = 0. \end{aligned}$$

- 2) Over the set $P_A \cap \{(s, q, p_0) : s < 0\}$, the vector field W points outward from the set P_A . In fact, if $p = p_0$, then

$$\begin{aligned} W_3 &= s(\lambda p_0 + 1 - \lambda^2 q) + p_0^3 \\ &\geq s(\lambda p_0 + 1) + p_0^3 = \frac{7}{8}s - \frac{1}{8^3 \lambda^3} \\ &\geq \frac{7}{8}s_0 - \frac{1}{8^3 \lambda^3} = \frac{-1}{8^4 \lambda^3} > 0. \end{aligned}$$

- 3) Over the set $P_A \cap \{(s, 0, p)\}$, the vector field W points outward from the set P_A . In fact, if $q = 0$, then

$$W_2 = sp \leq 0.$$

- 4) Over the set $P_A \cap \{(s, q_0, p)\}$, the vector field W points outward from the set P_A . In fact, if $q = q_0$, then

$$W_2 = sp - q_0(\lambda s - p^2) = (\lambda s - p^2) \left[\frac{sp}{\lambda s - p^2} - q_0 \right] \geq 0$$

because

$$\lambda s - p^2 < A s - p^2 \leq A s_0 - \frac{p_0^2}{4} = 0$$

and

$$h(s, p) = \frac{sp}{\lambda s - p^2} \leq h(s_0, p) \leq h(s_0, \frac{p_0}{2}) = \frac{1}{16\lambda^2} < q_0.$$

Thus, any orbit $\gamma(t)$ of W , with $\gamma(0)$ an interior point of P_A , has α -limit set contained in the line $s = p = 0$. Clearly, any (of these) orbit corresponds to an orbit of our initial vector field X that escapes to infinity. This completes the proof in this case.

Concerning assertion b), assume $n > 3$. The proof of the case $n = 3$ is a particular case of Theorem 3.9. Note that

$$F_{n,2}(x_1, \dots, x_n) = (F_{3,2}(x_1, x_2, x_3), \lambda x_4 + f(x_1, x_2)^3, \dots, \lambda x_n + f(x_1, x_2)^{n-1})$$

where $f(x_1, x_2) = x_2 - a x_1 - b x_1^2$. We have that the third iterate of $F_{n,2}$ is of the form

$$F_{n,2}^3(x_1, \dots, x_n) = (F_{3,2}^3(x_1, x_2, x_3), \lambda^3 x_4 + g_4(x_1, x_2, x_3), \dots, \lambda^3 x_n + g_n(x_1, x_2, x_3)).$$

The point $(\overline{x}_1, \dots, \overline{x}_n)$, where $(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ is a periodic point of period three of $F_{3,2}$ and $\overline{x}_j = \frac{1}{1-\lambda^3} g_j(\overline{x}_1, \overline{x}_2, \overline{x}_3)$, with $4 \leq j \leq n$, is a periodic point of period three of $F_{n,2}$, which completes the proof. \square

Note that $H_{3,2}$ has the special form

$$H_{3,2}(x, y, z) = (u(x, y), v(x, y, z), h(u(x, y))) . \quad (10)$$

By applying a linear change of coordinates to a large class of polynomial maps $H = (H_1, H_2, H_3)$ of the form

$$H(x, y, z) = (u(x, y, z), v(x, y, z), h(u(x, y, z), v(x, y, z))) \quad (11)$$

where JH is nilpotent such that H_1, H_2, H_3 are linearly independent, we obtain

$$G(x, y, z) = (g(t), v_1 z - (b_1 + 2v_1 \alpha x) g(t), \alpha g(t)^2) \quad (12)$$

with $t = y + b_1 x + v_1 \alpha x^2$ and $v_1 \alpha \neq 0$, and $g \in \mathbb{R}[t]$ with $g(0) = 0$ and $\deg_t g(t) \geq 1$. More specifically, we have the following.

Theorem 3.3. *Let $H(x, y, z) = (u(x, y, z), v(x, y, z), h(u(x, y, z), v(x, y, z)))$. Assume that $H(0) = 0$, $h'(0) = 0$, and the components of H are linearly independent over \mathbb{R} . Let $A = \frac{\partial v}{\partial x} \frac{\partial u}{\partial z} - \frac{\partial u}{\partial x} \frac{\partial v}{\partial z}$ and $B = \frac{\partial v}{\partial y} \frac{\partial u}{\partial z} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial z}$. If JH is nilpotent and $\deg_z(uA) \neq \deg_z(vB)$, then there exists a $T \in GL_3(\mathbb{R})$ such that THT^{-1} is of the form (12).*

(See [ChE].)

Remark 3.4. 1) Under the condition $\deg_z(uA) \neq \deg_z(vB)$, by Theorem 3.3, any vector field

$$X = \lambda I + (u(x, y, z), v(x, y, z), h(u(x, y, z), v(x, y, z))) \in \mathcal{N}_i(\lambda, 3)$$

has the form

$$X(x, y, z) = \lambda(x, y, z) + (0, v_1 z, 0) + g(t) (1, -(b_1 + 2v_1 \alpha x), \alpha g(t)) \quad (13)$$

up to a linear change of coordinates, where $t = y + b_1 x + v_1 \alpha x^2$, $v_1 \alpha \neq 0$, and $g \in \mathbb{R}[t]$, with $g(0) = 0$ and $\deg_t g(t) \geq 1$.

2) When $n = 3$, the vector field $F_{3,2}$ of Theorem 3.2 - - up to a linear change of coordinates - - has the form (13) with $g(t)$ a polynomial of degree one. Therefore, for this vector field both the MYC and the DMYC are false.

Consequently, we ask:

Question 2. Do there exist vector fields $X \in \mathcal{N}_i(\lambda, 3)$ of the form (13) for which the MY and/or the DMY Conjecture is true?

3.1 The continuous case

In the continuous case, our next result gives a negative answer to Question 2 when the degree of $g(t)$ is less than or equal to two. First note that by applying the coordinate change

$$\begin{aligned}(u, v, w) &= \phi(x, y, z) \\ &= (\lambda x + g(t), t, v_1 z + \lambda v_1 \alpha x^2)\end{aligned}$$

where $t = y + b_1 x + v_1 \alpha x^2$, to the vector field (13) we obtain

$$\phi_*(X)(u, v, w) = \lambda(u, v, w) + (g'(v)(\lambda v + w), w, \alpha u^2). \quad (14)$$

Theorem 3.5. *Consider a vector field $X \in \mathcal{N}_i(\lambda, 3)$, with $\lambda < 0$, of the form (13) where $g(t) = A_1 t + A_2 \frac{t^2}{2}$. Then X has orbits that escape to infinity.*

Proof. In the case $A_2 = 0$, making the linear change of coordinates

$$(u, v, w) = \phi(x, y, z) = \frac{1}{m}(x, my, mv_1 z)$$

where $m = A_1$, the vector field $X - \lambda I$ has the form (9). The result now follows from Theorem 3.2.

Next consider the case $A_2 \neq 0$. We may assume

$$X(x, y, z) = \lambda(x, y, z) + (g'(y)(\lambda y + z), z, v_1 \alpha x^2).$$

To find orbits of X which escape to infinity, we first make the coordinate change

$$(u, v, w) = \frac{1}{z}(x, y, 1).$$

If Y is the vector field X in the new coordinates, then $Z = wY$ is defined by

$$Z(u, v, w) = (-\beta u^3 + (A_1 w + A_2 v)(\lambda v + 1), -\beta u^2 v + w, -w(\lambda w + \beta u^2))$$

where $\beta = v_1 \alpha$. For $w \neq 0$, the vector fields Y and Z have the same orbits. Moreover, for $w > 0$ (resp. $w < 0$), the orbits of Y and Z have the same (resp. inverse) orientation. Now we apply the blow-up

$$(s, q, p) = (u, \frac{v}{u^3}, \frac{w}{u^5}).$$

If Y_1 is the vector field Y in the new coordinates, then $Y_1 = s^2 Z_1$ where

$$Z_1(s, q, p) = A(s, q, p) (s, -3q, -5p) + (0, p - \beta q, -p(\beta + \lambda p s^3))$$

and where $A(s, q, p) = -\beta + (A_1 p s^2 + A_2 q)(\lambda q s^3 + 1)$.

The singularities of Z_1 over $s = 0$ are

$$(0, 0, 0), \quad (0, \frac{2\beta}{3A_2}, 0), \quad \text{and} \quad (0, \frac{4\beta}{5A_2}, \frac{8\beta^2}{25A_2}).$$

The Jacobian matrix of Z_1 at $(0, \frac{4\beta}{5A_2}, \frac{8\beta^2}{25A_2})$ has eigenvalues

$$\mu_1 = -\frac{\beta}{5}, \quad \mu_2 = -\frac{2\beta}{5}, \quad \text{and} \quad \mu_3 = -2\beta.$$

In the case $\beta > 0$ (resp. $\beta < 0$), this singularity is an attractor (resp. repeller) for vector field Z_1 . Given an initial condition $(s(0), q(0), p(0))$ sufficiently close to the singularity, with $s(0)p(0) > 0$ (resp. $s(0)p(0) < 0$) for $\beta > 0$ (resp. $\beta < 0$), we obtain an orbit of the original vector field X that escapes positively to infinity.

□

3.2 The discrete case

In the discrete case, we prove that the answer to Question 2 is negative for any $g(t) \in \mathbb{R}[t]$ where $g(0) = 0$ and $\deg_t g(t) \geq 1$.

For $|\lambda| < 1$, consider

$$\begin{aligned} F(x, y, z) &= \lambda(x, y, z) + (0, v_1 z, 0) + \\ &\quad g(t)(1, -(b_1 + 2v_1 \alpha x), \alpha g(t)) \end{aligned} \tag{15}$$

where $t = y + b_1 x + v_1 \alpha x^2$, $v_1 \alpha \neq 0$, and $g(t) \in \mathbb{R}[t]$ with $g(0) = 0$ and $\deg_t g(t) \geq 1$.

Lemma 3.6. *The set of fixed points of F reduces to the origin.*

Proof. If (x_0, y_0, z_0) is a fixed point of F and $t_0 = y_0 + b_1 x_0 + v_1 \alpha x_0^2$, then we have

$$\begin{aligned} (1 - \lambda)x_0 &= g(t_0), \\ (1 - \lambda)z_0 &= \alpha g(t_0)^2, \text{ and} \\ (1 - \lambda)y_0 &= v_1 \alpha (1 - \lambda) x_0^2 - (b_1 + 2v_1 \alpha x_0)(1 - \lambda)x_0 \\ &= -b_1(1 - \lambda)x_0 - v_1 \alpha (1 - \lambda)x_0^2. \end{aligned}$$

Therefore,

$$t_0 = 0 \implies g(t_0) = 0 \implies (x_0, y_0, z_0) = (0, 0, 0).$$

□

Periodic points of period two. Assume $(x_0, y_0, z_0) \neq (0, 0, 0)$ is a periodic point of period two of F , and let $\beta = v_1 \alpha$. Then

$$C_1(x_0, y_0, z_0) = (\lambda^2 - 1)x_0 + \lambda g(t_0) + g(t_1) = 0, \quad (16)$$

$$C_2(x_0, y_0, z_0) = (\lambda^2 - 1)z_0 + \lambda \alpha g(t_0)^2 + \alpha g(t_1)^2 = 0, \quad (17)$$

$$C_3(x_0, y_0, z_0) = (\lambda^2 - 1)y_0 + 2\lambda v_1 z_0 - \lambda(b_1 + 2\beta x_0)g(t_0) + \beta g(t_0)^2 - (b_1 + 2\beta(\lambda x_0 + g(t_0)))g(t_1) = 0 \quad (18)$$

where

$$\begin{aligned} t_0 &= y_0 + b_1 x_0 + \beta x_0^2 \quad \text{and} \\ t_1 &= \lambda b_1 x_0 + \lambda y_0 + v_1 z_0 - 2\beta x_0 g(t_0) + \beta(\lambda x_0 + g(t_0))^2. \end{aligned}$$

Lemma 3.7. *If $g(t) = t$, then the unique periodic point of period two of F is the origin.*

Proof. Suppose $F^2(x_0, y_0, z_0) = (x_0, y_0, z_0)$. In this case, we have

$$C_1(x_0, y_0, z_0) = 0 \implies z_0 = \frac{1}{v_1} [(\lambda\beta + 1 - \lambda^2)x_0 - 2(\lambda - \beta)t_0 - \beta(\lambda x_0 + t_0)^2].$$

Replacing this value of z_0 in the equation $\frac{v_1}{1+\lambda} C_3(x_0, y_0, z_0) - C_2(x_0, y_0, z_0) = 0$, we obtain

$$(1 + \lambda)^2 (-x_0 + b_1 x_0 + \lambda x_0 + \beta x_0^2 + y_0) = 0.$$

Replacing our z_0 and the value $y_0 = (1 - \lambda - b_1)x_0 - \beta x_0^2$ in equations (16) through (18), we obtain

$$\begin{aligned} C_1(x_0, y_0, z_0) &= 0, \\ C_2(x_0, y_0, z_0) &= -(\lambda - 1)^3 x_0 = 0, \\ C_3(x_0, y_0, z_0) &= \frac{1}{v_1} (\lambda - 1)^3 (1 + \lambda) x_0 = 0. \end{aligned}$$

This implies that $(x_0, y_0, z_0) = (0, 0, 0)$, which completes the proof. □

Periodic points of period three. Assume $(x_0, y_0, z_0) \neq (0, 0, 0)$ is a periodic point of period three of F , and let $\beta = v_1 \alpha$. Then

$$\begin{aligned} D_1(x_0, y_0, z_0) &= (\lambda^3 - 1)x_0 + \lambda^2 g(t_0) + \lambda g(t_1) + g(t_2) = 0, \\ D_2(x_0, y_0, z_0) &= (\lambda^3 - 1)z_0 + \lambda^2 \alpha g(t_0)^2 + \lambda \alpha g(t_1)^2 + \alpha g(t_2)^2 = 0, \\ D_3(x_0, y_0, z_0) &= (\lambda^3 - 1)y_0 + 3\lambda^2 v_1 z_0 - \lambda^2 (b_1 + 2\beta x_0) g(t_0) + 2\lambda \beta g(t_0)^2 \\ &\quad - \lambda (b_1 + 2\beta (\lambda x_0 + g(t_0))) g(t_1) + \beta g(t_1)^2 \\ &\quad - (b_1 + 2\beta (\lambda^2 x_0 + \lambda g(t_0) + g(t_1))) g(t_2) = 0 \end{aligned}$$

where

$$\begin{aligned} t_0 &= y_0 + b_1 x_0 + \beta x_0^2 \quad \text{and} \\ t_1 &= \lambda b_1 x_0 + \lambda y_0 + v_1 z_0 - 2\beta x_0 g(t_0) + \beta (\lambda x_0 + g(t_0))^2, \\ t_2 &= b_1 \lambda^2 x_0 + \lambda^2 y_0 + 2\lambda v_1 z_0 - 2\beta \lambda x_0 g(t_0) + \beta g(t_0)^2 \\ &\quad - 2\beta (\lambda x_0 + g(t_0)) g(t_1) + \beta (\lambda^2 x_0 + \lambda g(t_0) + g(t_1))^2. \end{aligned}$$

Lemma 3.8. *If $-1 < \lambda < 1$ and $g(t) = At$, with $A \neq 0$, then F has a periodic point of period three $(x_0, y_0, z_0) \neq (0, 0, 0)$. Furthermore, the eigenvalues of $DF^3(x_0, y_0, z_0)$ are all other than 1.*

Proof. Computations were done using MATHEMATICA. These proved that the point (x_0, y_0, z_0) , where

$$\begin{aligned} x_0 &= \frac{(1 + \lambda + \lambda^2)(1 + 4\lambda^2 + \lambda^4)}{A\beta(1 - \lambda)^3}, \\ y_0 &= -\frac{1 + \lambda + \lambda^2}{A^2\beta(1 - \lambda)^6} [\lambda(1 + \lambda + \lambda^2)(4 + \lambda + 8\lambda^2 + 11\lambda^3 + 4\lambda^4 + 7\lambda^5 + \lambda^7) \\ &\quad + Ab_1(1 - \lambda)^3(1 + 4\lambda^2 + \lambda^4)], \quad \text{and} \\ z_0 &= \frac{(1 + \lambda + \lambda^2)^3(1 + 3\lambda^2 + 4\lambda^3 + 3\lambda^4 + \lambda^6)}{v_1 A^2 \beta (1 - \lambda)^5} \end{aligned}$$

is a periodic point of period three of F . They also proved that the characteristic polynomial of $DF^3(x_0, y_0, z_0)$ is

$$\begin{aligned} p(x) &= -\lambda^9 - \lambda(8 + 44\lambda + 104\lambda^2 + 164\lambda^3 + 164\lambda^4 + 113\lambda^5 + 44\lambda^6 \\ &\quad + 8\lambda^7 - 4\lambda^8)x + (-4 + 8\lambda + 44\lambda^2 + 113\lambda^3 + 164\lambda^4 + 164\lambda^5 \\ &\quad + 104\lambda^6 + 44\lambda^7 + 8\lambda^8)x^2 + x^3, \end{aligned}$$

and that

$$p(1) = 3(\lambda - 1)^3(1 + \lambda + \lambda^2)^3 \neq 0.$$

□

Theorem 3.9. *For $|\lambda| < 1$, consider*

$$\begin{aligned} F(x, y, z) = & \lambda(x, y, z) + (0, v_1 z, 0) + \\ & g(t)(1, -(b_1 + 2v_1 \alpha x), \alpha g(t)) \end{aligned}$$

where $t = y + b_1 x + v_1 \alpha x^2$, $v_1 \alpha \neq 0$, and $g(t) \in \mathbb{R}[t]$, with $g(0) = 0$ and $g'(0) \neq 0$. Then there exists $(x_0, y_0, z_0) \neq (0, 0, 0)$ which is a periodic point of period 3 of F .

Proof. Assume that $g(t) = At + A_2 t^2 + \dots + A_k t^k$, with $A \neq 0$. If $(A_2, \dots, A_k) = (0, \dots, 0)$, then F will be denoted F_0 . Therefore, F_0 has a periodic point of period three $(x_0, y_0, z_0) \neq (0, 0, 0)$ and the eigenvalues of $DF_0^3(x_0, y_0, z_0)$ are other than from 1. Consider the map $G : \mathbb{R}^{k-1} \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by

$$G(A_2, \dots, A_k, x, y, z) = F^3(x, y, z) - (x, y, z).$$

Note that $G(0, \dots, 0, x, y, z) = F_0^3(x, y, z) - (x, y, z)$, for all $(x, y, z) \in \mathbb{R}^3$. Then $G(0, \dots, 0, x_0, y_0, z_0) = (0, 0, 0)$ and $D_2 G(0, \dots, 0, x_0, y_0, z_0)$ is invertible. By the Implicit Function Theorem, there exists $\varepsilon > 0$ such that, for all (A_2, \dots, A_k) with $\max\{|A_2|, \dots, |A_k|\} < \varepsilon$, the map $F(A_2, \dots, A_k, \cdot, \cdot, \cdot)$ has a periodic point of period three. In the general case, note that if $a \in \mathbb{R} - \{0\}$ and $T(x, y, z) = a^{-1}(x, y, z)$, then

$$\begin{aligned} T(F(T^{-1}(u, v, w)) = & \lambda(u, v, w) + (0, v_1 w, 0) + \\ & \tilde{g}(t)(1, -(b_1 + 2v_1 \tilde{\alpha} u), \tilde{\alpha} \tilde{g}(t)) \end{aligned}$$

where $\tilde{\alpha} = \alpha a$ and

$$\tilde{g}(t) = a^{-1} g(at) = At + A_2 a t^2 + \dots + A_k a^{k-1} t^k.$$

For $|a|$ sufficiently small, the map $T \circ F \circ T^{-1}$ has a non-vanishing periodic point of period three and, consequently, so does F . □

References

- [ChE] M. Chamberland, A. van den Essen, *Nilpotent Jacobians in dimension three*, Journal of Pure and Applied Algebra **205** (2006), 146–155.
- [CEGMH] A. Cima, A. van den Essen, A. Gasull, E. Hubbers and F. Mañosas, *A Polynomial Counterexample to the Markus–Yamabe Conjecture*, Advances in Mathematics **131** (1997), 453–457.
- [CGM1] A. Cima, A. Gasull, F. Mañosas, *A Polynomial Class of Markus–Yamabe Counterexamples*, Publicacions Matemàtiques **41** (1997), 85–100.
- [CGM2] A. Cima, A. Gasull, F. Mañosas, *The discrete Markus–Yamabe problem*, Nonlinear Anal. **35** (1999), 343–354.
- [vE] A. van den Essen, *Polynomial Automorphisms and the Jacobian Conjecture*, Progress in Mathematics, vol. 190, Birkhauser, Basel, 2000.
- [Gui-Cast] V. Guíñez and A. Castañeda, *A Polynomial Class of Markus–Yamabe Counterexamples and Examples in \mathbb{R}^3* , Applicable Analysis **90** (2011), 787–798.
- [Gu] C. Gutierrez, *A solution to the bidimensional Global Asymptotic Stability Conjecture*, Ann. Inst. H. Poincaré Anal. Non Linéaire **12** (1995), 627–671.
- [MY] L. Markus, H. Yamabe, *Global Stability Criteria for Differential Systems*, Osaka Math. J. **12** (1960), 305–317.